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COMPARISON OF THE SECOND-ORDER TUNE SHIFT FORMULAS DUE TO SEXTUPOLES GIVEN BY COLLINS AND OHNUMA*

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COMPARISON OF THE SECOND-ORDER TUNE SHIFT FORMULAS DUE TO SEXTUPOLES GIVEN BY COLLINS AND OHNUMA

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Recently, T. Collins[1] put forth a theory of beam distortion and tune-shift due to sextupoles around the accelerator ring. He obtained for the second-order tune shifts the formulas

$$\Delta\nu_x = -(a^2/4\pi)\left\{\sum_k(B_3 s)_k + 3\sum_k(B_1 s)_k\right\}$$

$$-(b^2/2\pi)\left\{\sum_k(B_S\bar{s})_k + \sum_k(B_D\bar{s})_k - 2\sum_k(B_1\bar{s})_k\right\},$$

$$\Delta\nu_y = -(a^2/2\pi)\left\{\sum_k(B_S\bar{s})_k + \sum_k(B_D\bar{s})_k - 2\sum_k(B_1\bar{s})_k\right\}$$

$$-(b^2/4\pi)\left\{\sum_k(B_S\bar{s})_k - \sum_k(B_D\bar{s})_k + 4\sum_k(\bar{B}\bar{s})_k\right\}, \qquad (1)$$

where a and b are the envelopes of the horizontal and vertical amplitudes at a reference point where the horizontal beta-function is β_0. In above, B_1, B_3, $\bar{B}$, B_S and B_D are called distortion functions which can be computed easily at each location around the accelerator ring. The formula for one of them is given by Eq. (7) below. These B's are in fact lattice functions due to the presence of sextupoles just as the beta-functions are lattice functions due to the presence of quadrupoles. The quantities s and $\bar{s}$ are normalized sextupole strength (see Eq. (6) below) and the summations are taken over each of the sextupoles.

In a contribution to the Conference on the Intersections between Particle and Nuclear Physics at Steamboat Springs, Ohnuma[2] also computed the second-order tune-shifts due to sextupoles. His approach is totally different. He expanded the sextupole strength as harmonics around the ring and performed a canonical transformation so as to solve the equations of motion exactly up to first order in sextupole strength. The rest of the Hamiltonian that gives rise to the second-order tune-shift is

$$\Delta K = (2J_x)^2\cdot\left(\frac{9}{2}\right)\left\{\sum\frac{A_{3m}^2}{m-3\nu_x} + 3\sum\frac{A_{1m}^2}{m-\nu_x}\right\}$$

$$+(2J_y)^2\left(\frac{1}{2}\right)\left\{\sum\frac{4B_{1m}^2}{m-\nu_x} + \sum\frac{B_{\pm m}^2}{m-\nu_\pm}\right\}$$

$$+(2J_x)(2J_y)\cdot 2\left\{\sum\frac{B_{+m}^2}{m-\nu_+} - \sum\frac{B_{-m}^2}{m-\nu_-} - 6\sum\frac{A_{1m}B_{1m}}{m-\nu_x}\cos(\alpha_{1m}-\beta_{1m})\right\}.$$

Here, the summations are over the harmonic number m from $-\infty$ to $+\infty$. The A's and B's are the absolute values of the Fourier coefficients or harmonics of the sextupoles, each of which is in turn a series summation over the sextupoles, (see Eq.(4) below). A more complete explanation of the symbols can be found in Reference 2. From this Hamiltonian, the tune-shifts can be easily obtained from

$$\Delta\nu_x = \partial\Delta K/\partial J_x \quad \text{and} \quad \Delta\nu = \partial\Delta K/\partial J_y. \qquad (3)$$

These formulas appear to be quite different from those given by Collins in Eq. (1). The purpose of this note is to show that they are in fact exactly the same.

Let us consider the term $\sum_m A_{1m}^2/(m-\nu_x)$ in Ohnuma's Hamiltonian (2). The definition for this harmonic coefficient is

$$A_{1m}\, e^{i\alpha_{1m}} = \frac{i}{48\pi}\sum_k(S\beta_x^{3/2})_k \exp i(Q_x+m\theta)_k \qquad (4)$$

where A_{1m} and α_{1m} are real. The independent variable is θ_k = (distance along ring/average ring radius) which represents the location of the k^{th} sextupole of strength $S_k = (B''\ell/B\rho)_k$. The quantity $Q_{xk} = (\psi_x-\nu_x\theta)_k$ with ψ_{xk} denoting the horizontal Floquet phase of the sextupole. From Eq. (3), we get

$$A_{1m}^2 = (48\pi)^{-2}\sum_{kk'}(S\beta_x^{\frac{3}{2}})_k(S\beta_x^{\frac{3}{2}})_{k'}\exp i[m(\theta_k-\theta_{k'}) + (Q_k-Q_{k'})].$$

The summation over m can be accomplished using the formula

$$\sum_{m=-\infty}^{\infty}\frac{e^{i(m\theta+b)}}{m-\nu} = \begin{cases} -\dfrac{\pi}{\sin\pi\nu}\,e^{-i[\nu(\pi-\theta)-b]} & 0<\theta<2\pi \\ -\pi\cot\pi\nu\; e^{ib} & \theta=0. \end{cases}$$

Therefore,

$$\sum_{m=-\infty}^{\infty}\exp i[m(\theta_k-\theta_{k'}) + (Q_k-Q_{k'})]/(m-\nu_x)$$

$$= \begin{cases} -\pi\cot\pi\nu_x & \theta_k=\theta_{k'} \\ -(\pi/\sin\pi\nu_x)\exp i(\psi_{xk}-\psi_{xk'}\pm\pi\nu_x) & \theta_k\lessgtr\theta_{k'}. \end{cases}$$

Noting that $\psi_{xk}\gtrless\psi_{xk'}$ when $\theta_k\gtrless\theta_{k'}$, by combining the above, we get

$$\sum_{m=-\infty}^{\infty}\frac{A_{1m}^2}{m-\nu_x} = -\frac{\pi}{(48\pi)^2\sin\pi\nu_x}$$

$$\sum_{kk'}(S\beta_x^{\frac{3}{2}})_k(S\beta_x^{\frac{3}{2}})_{k'}\cos(|\psi_{xk}-\psi_{xk'}|-\pi\nu_x). \qquad (5)$$

Collins normalized the sextupole strength as

$$s_k = \frac{1}{2}S_k(\beta_x^3/\beta_0)_k^{1/2} \quad \text{and} \quad \bar{s}_k = \frac{1}{2}S_k(\beta_x\beta_y^2/\beta_0)_k^{1/2} \qquad (6)$$

so that all the betas are absorbed. In Eq. (6), S_k is

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the same sextupole strength defined by Ohnuma. Note also the factor of $\frac{1}{2}$. One of Collin's distortion functions at location ψ_{xk} is defined as

$$B_{1k} = \frac{1}{8\sin\pi\nu_x} \sum_{k'} s_k \cos(|\psi_{xk}-\psi_{xk'}|-\pi\nu_x). \qquad (7)$$

With these notations, the summation (5) becomes

$$\sum_{m=-\infty}^{\infty} \frac{A_{1m}^2}{m-\nu_x} = -(\beta_o/72\pi) \sum_k (B_1 s)_k. \qquad (8)$$

In exactly the same way,

$$\sum_{m=-\infty}^{\infty} \frac{A_{3m}^2}{m-3\nu_x} = -(\beta_o/72\pi) \sum_k (B_3 s)_k, \qquad (9)$$

$$\sum_{m=-\infty}^{\infty} \frac{B_{1m}^2}{m-\nu_x} = -(\beta_o/8\pi) \sum_k (\bar{B}\bar{s})_k, \qquad (10)$$

$$\sum_{m=-\infty}^{\infty} \frac{B_{\pm m}^2}{m-\nu_\pm} = \mp(\beta_o/8\pi) \sum_k (B_{S,D}\bar{s})_k, \qquad (11)$$

$$\sum_{m=-\infty}^{\infty} \frac{A_{1m}B_{1m}\cos(\alpha_{1m}-\beta_{1m})}{m-\nu_x} = -(\beta_o/24\pi) \sum_k (\bar{B}s)_k. \qquad (12)$$

The change in sign in Eq. (11) comes about because Collins used $\sin\pi(2\nu_y-\nu_x)$ in his definition of B_D whereas Ohnuma used the definition $\nu_- = \nu_x-2\nu_y$. The factor in front of the summations in Eqs. (10) and (11) are 9 times bigger than those in Eqs. (8) and (9) because in the definition of A_{1m} and A_{3m} Ohnuma included the factor $i/48\pi$ whereas for B_{1m} and $B_{\pm m}$, the corresponding factor was $i/16\pi$.

Substituting Eqs. (8) - (12) into the Hamiltonian (2), we finally get

$$\Delta K = -(J_x^2\beta_o/4\pi)\left[\sum_k (B_3 s)_k + 3\sum_k (B_1 s)_k\right]$$

$$-(J_y^2\beta_o/4\pi)\left[4\sum_k (\bar{B}\bar{s})_k + \sum (B_S\bar{s})_k - \sum_k (B_D\bar{s})_k\right]$$

$$-(J_x J_y \beta_o/\pi)\left[\sum_k (B_S\bar{s})_k + \sum_k (B_D\bar{s})_k - 2\sum_k (B_1\bar{s})_k\right]$$

Using the Hamiltonian equations of motion (3) and the fact that the envelopes of the horizontal and vertical amplitudes at the particular reference point are given by

$$a = (2J_x\beta_o)^{\frac{1}{2}} \quad \text{and} \quad b = (2J_y\beta_o)^{\frac{1}{2}}$$

we reproduce exactly the same formulas for $\Delta\nu_x$ and $\Delta\nu_y$ as given by Collins in Eq. (1).

References

1. T. Collins, Proceedings of the 1984 Summer Study on the Design and Utilization of the Superconducting Super Collider, Snowmass, Colorado, 1984

2. S. Ohnuma, Proceedings of the Conference on the Interactions between Particle and Nuclear Physics, Steamboat Springs, Colorado, 1984.